MATHEMATICS

HOMOGENEOUS RIEMANNIAN MANIFOLDS OF NON-POSITIVE SECTIONAL CURVATURE

BY

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I. *Introduction*

We will prove the following result:

Theorem A. *Every connected homogeneous Riemannian manifold M of non-positive sectional curvature is either itself diffeomorphic to a vector bundle on a connected solvable group, or a covering M' of M has this property, where the group of covering transformations is a finite abelian group all of whose elements are of order two.*

As background, recall that J. WOLF has proved that every homogeneous Riemannian manifold of constant negative curvature is simply connected [9]. S. KOBAYASHI has proved [6] that every homogeneous Riemannian manifold of non-positive sectional curvature and negative definite Ricci curvature is simply connected. I am indebted to Professor Kobayashi for mentioning his theorem to me, since it suggested the above theorem.

The proof of Theorem A involves a combination of previous results on the critical points of the length function of a Killing vector field on a Riemannian manifold of non-positive sectional curvature [4] and an extension of known material [1, 2] on the calculus of variations on Riemannian manifolds. Our main result on the latter theory is the following theorem, generalizing the classical theorem of HADAMARD and CARTAN on Riemannian manifolds of non-positive curvature [2].

Theorem B. *Let M ne a complete Riemannian manifold and let N be connected, closed totally geodesic submanifold of M. Let $N^{\perp}$ be the normal bundle to N and let $Exp\colon N^{\perp} \to M$ be the usual exponential map of Riemannian geometry. If: a) the sectional curvature is non-positive for all tangent planes of M that contain tangent vectors to geodesics of M that are perpendicular at one point to M and b) the image of $\pi_1(N)$ under the inclusion map is onto $\pi_1(M)$, then Exp is a diffeomorphism.*

2. *Preliminaries from Riemannian geometry*

Suppose that M is a complete Riemannian manifold [1]. We will suppose, unless mentioned otherwise, that curves are parameterized proportionally to arc length. Let $\sigma\colon [0, 1] \to M$ be a curve in M and let $v\colon t \to v(t) \in M_{\sigma(t)}$, $0 \leqslant t \leqslant 1$, be a vector field along σ. The usual Levi–Civita affine connection associated with the Riemannian metric [1, 7] defines the *covariant derivative vector field of* v, denoted by ∇v, as another vector field along σ. The precise definition of ∇v is given in [1] or [3]. If v arises by restricting a vector field X on M to σ, i.e. if $v(t) = X(\sigma(t))$ for $0 \leqslant t \leqslant 1$, then $\nabla v(t) = \nabla_{\sigma'(t)} X$, the covariant derivative of X in the direction $\sigma'(t)$. Recall also that $\nabla v = 0$ is the condition that $v(t)$ be the parallel translate along σ of $v(0)$, for $0 \leqslant t \leqslant 1$.

Suppose that $\delta(s, t)$, $0 \leqslant s, t \leqslant 1$, is a homotopy of curves in M. t will be regarded as the parameter of the curves, s as the deformation parameter. We recall some definitions given in [3]: $D_t\delta(s, t)$ $(D_s\delta(s, t))$ is the tangent vector to the curve $u \to \delta(s, t+u)$ $(u \to \delta(s+u, t)$ at $u = 0$. If $v(s, t) \in M_{\delta(s,t)}$, $0 \leqslant s, t \leqslant 1$, is a vector field along δ, $\nabla_t v(s, t)$ $(\nabla_s v(s, t))$ is the covariant derivative at $u = 0$ of the vector field $u \to v(s, t+u)$ $(u \to v(s+u, t))$ along the curve $u \to \delta(s, t+u)$ $(u \to \delta(s+u, t))$. The following relations are derived in a straightforward way from the Cartan Structure Equations for Riemannian geometry

2.1. a) $\nabla_t D_s \delta(s, t) = \nabla_s D_t \delta(s, t)$

b) $\nabla_t \nabla_s v(s, t) - \nabla_s \nabla_t v(s, t)$
$= R(D_t(s, t), D_s(s, t))(v(s, t)$

where $R(\ ,\ )(\ )$ *is the Riemann curvature tensor of the Riemannian manifold.*

As an illustration of the application of this formalism, we derive the differential equations of a Jacobi field. Suppose that, $t \to \delta(s, t)$ is a geodesic of M for each s, i.e. δ is a geodesic deformation. Then,

$$\nabla_t D_t \delta(s, t) = 0.$$

Hence,

$$\begin{aligned} 0 = \nabla_s \nabla_t \delta(s, t) &= \nabla_t \nabla_s D_t \delta(s, t) + R(D_s\delta, D_t\delta)\ (D_t\delta) \\ &= \nabla_t \nabla_t D_s \delta(s, t) + R(D_s\delta, D_t\delta)\ (D_t\delta). \end{aligned}$$

Suppose now that $\sigma(t) = \delta(0, t)$ is the initial curve of the homotopy. The

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[1] We shall assume known such standard facts of Riemannian geometry as the notion of completeness, the Hopf–Rinow theorem, the definition of geodesics, etc. All manifolds, tensor-fields, maps, curves, etc. will be of differentiability class C^∞ unless mentioned otherwise. If M is a manifold, if x is a point of M, M_x is the tangent space to M at x. If $\varphi\colon M \to M'$ is a map of manifolds, $\varphi_*\colon M_x \to M'_{\varphi(x)}$ is the linear map φ defines on tangent vectors. If $\sigma\colon [0, b] \to M$ is a curve, $\sigma'(t) \in M_{\sigma(t)}$, for $a \leqslant t \leqslant b$, denotes the tangent vector to σ at t. If M is Riemannian, $<, >$ denotes the inner product on tangent vectors.

vector field $v(t) = D_s\delta(s, t)$ along σ will be called the *infinitesimal deformation* of σ corresponding to the homotopy (or deformation) δ. We then have:

2.2. $\nabla\nabla v(t) = R(\sigma'(t), v(t))\,(\sigma'(t))$, i.e. *$v$ satisfies the Jacobi equations.*

A vector field v along σ is called a *Jacobi field* if it satisfies 2.2. Thus, we have shown that the infinitesimal deformation of a geodesic deformation is a Jacobi field. Conversely, each such Jacobi field along σ arises in this way from at least one such geodesic deformation. Suppose that N is a submanifold of M such that $\sigma(0) \in N$ and $\sigma'(0)$ is perpendicular to $N_{\sigma(0)}$, the tangent space to N at $\sigma(0)$. A Jacobi field $v(t) \in M_{\sigma(t)}$ will be said to be *transversal* to N if there is at least one geodesic deformation $\delta(s, t)$ of σ such that: a) $\delta(s, 0) \in N$ for $0 \leqslant s \leqslant 1$, b) $t \to \delta(s, t)$ is a geodesic of M that is perpendicular to N at $t = 0$, for each s. In reality, the definition of transversality can be made independent of the existence of such a deformation, requiring only that $v(0)$ and $\nabla v(0)$ satisfy certain linear conditions. We shall be interested in these conditions only in case N is a totally geodesic submanifold. Then, the vector field $s \to D_t\delta(s, 0)$ is perpendicular to N, hence so is $\nabla_s D_t\delta(s, 0) = \nabla_t D_s\delta(s, 0)$, hence so is $\nabla v(0)$, while $v(0)$ is tangent to N. Thus we have the conditions of transversality:

2.3. $v(0) \in N_{\sigma(0)}$, $\nabla v(0) \in N^{\perp}{}_{\sigma(0)}$ *in case N is totally geodesic.*

In particular, we see that the space of Jacobi vector fields transversal to σ is equal to the dimension of M. (This is true for any submanifold N of M, as a matter of fact.) Let N be the normal bundle to N, defined as the set of tangent vectors v to points x of N such that $v \in N^{\perp}_x$, i.e. such that v is perpendicular to N_x. Let Exp: $N^{\perp} \to M$ be the exponential map. Recall that, for each $v \in N^{\perp}_x$, $x \in N$, Exp $(v) = \sigma(1)$, where $\sigma: [0,1] \to M$ is the unique geodesic of M such that $\sigma(0) = x$ and $\sigma'(0) = v$. Thus, for $v \in M_x$, $t \to$ Exp (tv) is the geodesic of M starting at x and tangent there to v. If $(\gamma(s), u(s))$, $\gamma(s) \in N$, $u(s) \in N^{\perp}{}_{\sigma(s)}$, $0 \leqslant s \leqslant 1$, is a curve in $N^{\perp}$, and if $S \to$ Exp $(u(s))$ is the image of the curve in M under the exponential map, then Exp $(u(s)) = \delta(s, 1)$, where $\delta(s, t)$ is the geodesic homotopy of such that $\delta(s, 0) = \gamma(s)$, $D_t\delta(s, 0) = u(s)$. If $v(t) = D_s\delta(0, t)$ is the vector field along the curve $t \to$ Exp $(u(0)t) = \sigma(t)$ that is the infinitesimal deformation corresponding to δ, then $v(1)$ is the image under Exp_* of the tangent vector to the curve $(\gamma(s), u(s))$ in $N^{\perp}$ at $s = 0$. Thus we see that, given a $u \in N^{\perp}_x$, $x \in N$, Exp_*: $(N^{\perp})_u \to M_{\mathrm{Exp}(u)}$ is one-one if and only if there is no non-zero Jacobi vector field $v(t)$ along the curve $t \to$ Exp (ut) that is transversal to N and satisfies: $v(1) = 0$. If there is such a Jacobi field Exp (u) is said to be a *focal point* of N along the geodesic $t \to$ Exp (tu). We are mainly interested in this paper in the case where the absence of focal points can be predicted from curvature conditions. To derive these conditions, we turn to the description of how the length of a Jacobi field

v varies with t:

$$\frac{d}{dt}\langle v(t), v(t)\rangle = 2\langle \nabla v(t), v(t)\rangle$$

$$\frac{1}{2}\frac{d^2}{dt^2}\langle v(t), v(t)\rangle = \langle \nabla^2 v(t), v(t)\rangle + \langle \nabla v(t), \nabla v(t)\rangle$$

$$= \langle R(\sigma'(t), v(t))\,(\sigma'(t)), v(t)\rangle + \langle \nabla v(t), \nabla v(t)\rangle.$$

Now, if u_1, u_2 are tangent vectors to a point x of M, let $K(u_1, u_2)$ be the Riemann sectional curvature of the plane spanned by u_1 and u_2. If u_1 and u_2 are unit orthogonal vectors, recall that $K(u_1, u_2) = -\langle R(u_1, u_2)\,(u_1), u_2\rangle$. If v is a Jacobi vector field along a geodesic σ, v can be decomposed into the sum of two Jacobi fields, one perpendicular, one parallel to σ. Now, if v is parallel to σ, the growth of $\langle v, v\rangle$ is easily predicted: $v(t) = (at+b)\sigma'(t)$ for some constants a and b. Hence, it essentially suffices to only consider the case when v is perpendicular to σ'. Then, the above formulas take the form:

2.4 $$\frac{1}{2}\frac{d^2}{dt^2}\langle v(t), v(t)\rangle = -\langle v(t, v(t)\rangle\langle \sigma'(t), \sigma'(t)\rangle K\sigma'(t), v(t)) + \langle \Delta v(t), \Delta v(t)\rangle.$$

Using these formulas (due to E. Cartan [2]), we derive the following result:

Lemma 2.1. *Suppose σ: $[0, 1] \to M$ is a geodesic of M and $v: t \to \to v(t) \in M_{\sigma(t)}$ is a Jacobi vector field along such that $\langle v(0), \nabla v(0)\rangle = 0$. Suppose in addition that the sectional curvature of all tangent planes containing a tangent vector to σ is non-positive. Then, $\langle v(t), v(t)\rangle \geqslant \langle v(0), v(0)\rangle$ for all t.*

Proof. If v is tangent to σ this is obvious. It then suffices to deal with the case where it is perpendicular to σ. We have $d/dt\,\langle v(t), v(t)\rangle|_{t=0} = 0$. From 2.4 $d^2/dt^2\,\langle v(t), v(t)\rangle \geqslant 0$, hence $d/dt\,\langle v(t), v(t)\rangle \geqslant 0$, hence $t \to \langle v(t), v(t)\rangle$ is montotone increasing.

Theorem 2.2. *Suppose that N is a connected totally geodesic submanifold of M that is, as a set of points, closed in M. Suppose that the sectional curvature is non-positive for all tangent planes to M that contain a tangent vector of a geodesic of M that is perpendicular to N. Let $N^\perp$ be the normal bundle to N. Then, the map Exp: $N^\perp \to M$ is onto, and it is a diffeomorphism if and only if the induced map Exp_*: $\pi_1(N^\perp) \to \pi_1(M)$ of fundamental groups is onto.*

Proof. The Hopf–Rinow theorem and the fact that N is closed in M implies that Exp $(N^\perp) = M$. It follows from Lemma 2.1 that N has no focal points, hence that the map Exp is everywhere of maximal rank, hence Exp is a local diffeomorphism. We now try to lift curves in M by Exp.

Lemma 2.3. *Suppose that* $\alpha(s)$, $0 \leqslant s \leqslant 1$, *is a curve of* M *and that* $x_0 \in N$, $u_0 \in N^{\perp}_{x_0}$ *are such that* $\operatorname{Exp}(u_0) = \alpha(0)$. *Then, there are unique curves* $\gamma(s) \in N$, $u(s) \in N^{\perp}_{\gamma(s)}$, $0 \leqslant s \leqslant 1$, *so that* $Exp\,(u(s)) = \alpha(s)$, $\gamma(0) = x_0$ *and* $u(0) = u_0$, *i.e. the curve* α *can be lifted back to* $N^{\perp}$ *via Exp. Further, the length of* $\gamma(s)$, $0 \leqslant s \leqslant 1$, *is no greater than the length of* α, *and* $\langle u(s), u(s)\rangle$ *is uniformly bounded in* s *by a bound depending only on the length of* α *and on* $\langle u(0), u(0)\rangle$.

It should be clear that the lifting of α is unique, since Exp is everywhere of maximal rank. Now, if α is sufficiently small, it has a lifting by Exp. Suppose that we try to lift α globally by continuation: This process will succeed unless an obstacle is encountered in the form of a number b such that $\alpha(s)$ can be lifted over $0 \leqslant s < b$ but over no larger interval. Suppose then that $\gamma(s)$ and $u(s)$ satisfying the conditions of the Lemma exist over $0 \leqslant s < b$. Let $\delta(s, t) = \operatorname{Exp}(tu(s))$, $0 \leqslant t \leqslant 1$, $0 \leqslant s \leqslant b$. For each s, $t \to \delta(s, t)$ is a geodesic, and $D_t\delta(s, 0) = u(s)$. Then, $\delta(s, 0) = \gamma(s)$, hence $D_s\delta(s, 0) = \gamma'(s)$. From Theorem 2.1 and the fact that N is totally geodesic (hence all Jacobi fields v transverse to N satisfy $\langle \nabla v(0), v(0)\rangle = 0$) we have:

$$\langle D_s\delta(s, t), D_s\delta(s, t)\rangle \geqslant \langle D_s\delta(s, 0), D_s\delta(s, 0)\rangle, \text{ hence}$$
$$\langle \alpha'(s), \alpha'(s)\rangle \geqslant \langle \gamma'(s), \gamma'(s)\rangle \text{ for } 0 \leqslant s < b.$$

Thus, we see that the length of the curve $s \to \gamma(s)$, $0 \leqslant s < b$, is bounded by the length of α, hence $\lim \gamma(s)$ exists and equals, say x. But x must belong to N since N is closed in M. To deal with $u(s)$ as $s \to b$ proceed as follows:

$$\begin{aligned}\frac{d}{ds}\langle u(s), u(s)\rangle &= \frac{\partial}{\partial s}\langle D_t\delta(s, t), D_t\delta(s, t)\rangle \\ &= 2\langle \nabla_t D_s\delta(s, t), D_t\delta(s, t)\rangle \\ &= 2\frac{\partial}{\partial t}\langle D_s\delta(s, t), D_t\delta(s, t)\rangle \text{ since } \nabla_t D_t\delta(s, t) = 0.\end{aligned}$$

Integrate both sides with respect to t over $0 \leqslant t \leqslant 1$:

$$\frac{d}{ds}\langle u(s), u(s)\rangle = 2\left.\langle D_s\delta(s, t), D_t\delta(s, t)\rangle\right|_{t=0}^{t=1}, \text{ hence}$$

$$\frac{d}{ds}\langle u(s), u(s)\rangle^{1/2} = \frac{\left.\langle D_s\delta(s, t), D_t\delta(s, t)\rangle\right|_{t=0}^{t=1}}{\langle u(s), u(s)\rangle^{1/2}}, \text{ hence}$$

$$\langle u(s), u(s)\rangle^{1/2} = \langle u(0), u(0)\rangle^{1/2} + \int_0^s \frac{\left.\langle D_s\delta(k, t), D_t\delta(k, t)\rangle\right|_{t=0}^{t=1}}{\langle u(k), u(k)\rangle^{1/2}}\,dk.$$

Now, $\langle D_t\delta(k, t), D_t\delta(k, t)\rangle = \langle u(k)\rangle$. Using this fact and the Schwarz inequality, we have:

$$\langle u(s), u(s)\alpha^{1/2} \leqslant \langle u(0), u(0)\rangle^{1/2} +$$

$$\int_0^s [\langle D_x\delta(k, 1), D_s\delta(k, 1)\rangle^{1/2} + \langle D_s\delta(k, 0), D_s\delta(k, 0)\rangle^{1/2}]\, dk$$

$$\leqslant \langle u(0), u(0)\rangle^{1/2} + 2\int_0^s \langle D_s(k, 1), D_s(k, 1)\rangle^{1/2}\, dk$$

$$\leqslant \langle u(0), u(0)\rangle^{1/2} + 2\ (\text{length of } \alpha(s),\ 0 \leqslant s \leqslant 1).$$

This shows that $\langle u(s), u(s)\rangle$ is bounded. Then, a sequence $(s_j\colon 1 \leqslant j \leqslant \infty)$ can be found so that $s_j \to b$ and $u(s_j) \to u \in N_x^{\perp}$ as $j \to \infty$. For j sufficiently large, $(\gamma(s_j), u(s_j))$ lies in a neighborhood of (x, u) in $N^{\perp}$ so that Exp is a local diffeomorphism on this neighborhood. Then, both $\gamma(s)$ and $u(s)$ can be continued at least a little way beyond b, hence ultimately over all of $0 \leqslant s \leqslant 1$.

Return to the proof of the theorem.

At this point, if we could conclude that Exp were a covering map, we would be through. We do not know this [1], but can make the necessary modifications in the standard arguments. Suppose then that $\mathrm{Exp}_*(\pi_1(N^{\perp})) = \pi_1(M)$. First, we show that Exp is one-one. Suppose otherwise: Let (x, u), $(x_1, u_1) \in N^{\perp}$, $x, x_1 \in N$, $u \in N_x^{\perp}$, $u_1 \in N_{x_1}^{\perp}$, be points of $N^{\perp}$ such that $\mathrm{Exp}\,(u) = \mathrm{Exp}\,(u_1)$. Let $(x(k), u(k))$, $0 \leqslant b \leqslant 1$, be a smooth curve going from (x, u) to (x_1, u_1) in $N^{\perp}$. The image under Exp is then a closed curve in M based at $\mathrm{Exp}\,(u)$, say $y(k) = \mathrm{Exp}\,(u(k))$. If $y(k, s)$, $0 \leqslant s \leqslant 1$, is a smooth homotopy of closed curves in M, since the length of each curve $s \to y(k, s)$ is bounded, the homotopy can be lifted to a homotopy $(x(k, s), u(k, s))$ in $N^{\perp}$ such that $\mathrm{Exp}\,(u(k, s)) = y(k, s)$. Further, because of the bound we have derived in the Lemma on the length of $s \to x(k, s)$ and $\langle u(k, s)\rangle$, there is no difficulty in showing that $(x(k, s), u(k, s))$ can be chosen so as to be simultaneously continuous in k and s. In particular, since Exp is a local diffeomorphism, each of the curves $k \to (x(k,s), u(k,s))$ must have the same end-points, that is, the end-points must be (x, u) and (x_1, u_1). But, if $\mathrm{Exp}_*\,(\pi, (N^{\perp})) = \pi_1(M)$, there must be a closed curve in $N^{\perp}$ based at (x, u) whose projection under Exp is homotopic, via a smooth homotopy, to the closed curve $k \to y(k)$. This is only possible then if $(x, u) = (x_1, u_1)$, i.e. if Exp is one-one. It is now obvious that Exp is a diffeomorphism, since it is of maximal rank everywhere.

Theorem 2.3. *Suppose that N is a connected closed submanifold of TM. Then, any one of the following conditions is sufficient to guarantee that $Exp_*\,(\pi_1(N^{\perp})) = \pi_1(M)$.*

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[1]) It seems to be unknown, whether maps satisfying this sort of differential geometric condition are actually covering maps.

a) *The inclusion map $N \to M$ maps the fundamental group of N onto the fundamental group of M.*

b) *N has a geodesic tubular neighborhood U whose radius is bounded away from zero and such that U is a retract of M.*

c) *N has a tubular neighborhood U as in b), and further there is a real-valued C^∞ function f on M bounded from above whose set of critical points is contained in N and such that:*

$$\sup_{x \in M-U} f(x) < \sup_{x \in U} f(x).$$

Proof. Since N can be considered as included in $N^\perp$ as the 0-cross section, a) is obvious. b) follows from a), since U then has the same homotopy type as N.

To prove c), let grad f be the gradient vector field of f. Recall that grad f is defined by the property: $\langle \operatorname{grad} f, v\rangle = df(v) = v(f)$ for each tangent vector v to M. A curve σ: $[0, b] \to M$ is an integral curve for grad f if $\sigma'(t) = \operatorname{grad} f\,(\sigma(t))$ for $0 \leqslant t < b$. Then,

$$\begin{aligned}\frac{d}{dt} f(\sigma(t)) = df(\sigma'(t)) = \langle \operatorname{grad} f,\ \sigma'(t)\rangle &= \langle \sigma'(t), \sigma'(t)\rangle \\ &= \langle \operatorname{grad} f, \operatorname{grad} f\rangle(\sigma(t)).\end{aligned}$$

Now, we must show that a closed curve of M based at a point of N can be deformed so as to completely lie in U. $\langle(\operatorname{grad} f)(x), (\operatorname{grad} f)(x)\rangle > 0$ if $x \in M - U$, since all the critical points of f, i.e. the zeros of grad f, lie in N. If the minimum value of f on this closed curve is greater than the sup of g on $M - U$ we are through, since the closed curve would then have to lie in U. Suppose then that we deform the closed curve along the integral curves of grad f: The values of f at points outside of U must steadily increase, hence after a finite length of deformation time the deformed closed curve must satisfy the condition that the minimum of f on it is greater than to sup of f on $M - U$, hence it must lie in U.

3. *Proof of Theorem A*

Lemma 3.1. *Let M be a complete Riemannian manifold of non-positive sectional curvature and suppose that G is a closed, connected group of isometries of M that acts effectively. Suppose that $x_0 \in M$ is such that the isotropy group at x_0 is finite, and that the orbit of G at x_0, denoted by Gx_0, is totally geodesic. Then, G is solvable.*

Proof. Since Gx_0 is totally geodesic, it has non-positive sectional curvature also. Then, it suffices to deal with the case where G acts transitively on M. If G were not solvable, it would have a semi-simple connected compact subgroup H. The simply connected covering group H' of H, again compact, would act on M', the simply connected covering space of M, hence would have a fixed point in M', hence H would have a fixed point in M, contradicting that K is finite.

Lemma 3.2. *Suppose that M is a complete Riemannian manifold of non-positive sectional curvature and that X is a Killing vector field on M. Let f be the length function of X, i.e. $f(x)=\langle X(x), X(x)\rangle$ for $x \in M$. Then, the set of all absolute minima of f is a connected, totally geodesic submanifold N of M. f is constant on N. The orbit of X at a point $x \in M$ is a geodesic of M if and only if $x \in N$, and f has no critical points outside of N.*

This is proved in [4].

Lemma 3.3. *Let M be a Riemannian manifold, and let G be a closed, connected group of isometries of M. For $x \in M$, let Gx be the orbit of G at x. Then Gx has a geodesic tubular neighborhood in M whose radius is bounded away from zero.*

Proved in [8].

Lemma 3.4. *Let M be a Riemannian manifold and suppose that a connected Lie group G acts transitively on M as a group of isometries. Let $\mathbf{G}$ be the Lie algebra of G, considered as a Lie algebra of vector fields on M. Then, for $X \in \mathbf{G}$, each connected component N of the set of singular points of X is a totally geodesic submanifold of M, and a subgroup G_1 of G acts transitively on it. G_1 can be taken as the connected component of the group of all $g \in G$ such that $\operatorname{Ad} g(X)=X$, i.e. the Lie algebra $\mathbf{G}_1$ of G_1 is the centralizer of X in $\mathbf{G}$.*

Proof. The following facts are proven in [5]: N is totally geodesic, and, for $x \in N$, $N_x=\{v \in M_x\colon \nabla_v X=0\}$. Let K be the isotropy subgroup of G at $x \in N$. Since K is compact, there is a subspace $\mathbf{M} \subset \mathbf{G}$ such that $\mathbf{G}=\mathbf{K} \oplus \mathbf{M}$ and $\operatorname{Ad} K(\mathbf{M}) \subset \mathbf{M}$. Now, for each $v \in M_x$, there is a $Y \in \mathbf{M}$ with $Y(x)=v$. Since $X(x)=0$, it is easily seen by computing in local coordinates that $\nabla_v X=[Y, X](x)$. Thus, $\nabla_v X=0$ if and only if $[Y,X](x)=0$. Since $[Y, X] \in \mathbf{M}$, this is so if and only if $[Y, X]=0$, i.e. Y belongs to the centralizer of X in $\mathbf{G}$. This proves that the Lie subgroup of G corresponding to this centralizer acts transitively on N.

Lemma 3.5. *Let M be a Riemannian manifold with a connected transitive Lie group of isometries G and with non-positive sectional curvature. For each $x \in M$, there is connected, totally geodesic submanifold N passing through x and a subgroup S of G transitive on N such that:*

a) *The isotropy subgroup of S at each point of N is finite.*

b) *The inclusion map maps $\pi_1(N)$ onto $\pi_1(M)$.*

Proof. Let K be the isotropy subgroup of G at x. If K is finite, we are through. If not, there is at least one $X \in \mathbf{K}$ that is not identically zero, but of course $X(x)=0$. Let N_1 be the set of critical points of X. By Lemmas 3.2, 3.3, and Theorem 2.3, N_1 is a connected totally geodesic submanifold of M such that a subgroup G_1 of G acts transitively on it and such that the inclusion map sends $\pi_1(N_1)$ onto $\pi_1(M)$. If G_1 acts on

N_1 with finite isotropy group, we are through. If not, there is an $X_1 \in \mathbf{G}_1$ such that $X_1(x)=0$, N_2 can be taken as the set of $y \in N_1$ with $X_1(y)=0$, etc. We obtain on the way a decreasing sequence $N_1 \supset N_2 \supset \ldots$ of totally geodesic submanifolds. The last one can be taken as N.

Lemma 3.6. *Let S be a solvable connected Lie group and let $N=S/F$ be a homogeneous space of S, with F a discrete closed subgroup of S. Suppose further that N has a Riemannian metric invariant under S and that S acts effectively on N. Then, F is a finite abelian subgroup of S all of whose elements have order two.*

Proof. Let $\mathbf{S}$ be the Lie algebra of S, and let Ad be the adjoint representation of S in $\mathbf{S}$. The linear isotropy subgroup of S at a point of N is a faithful representation of F, since S acts effectively, and can be identified with Ad F. Since $\mathbf{S}$ is solvable, the elements of Ad S can be put simultaneously in triangular form by choice of a basis of $\mathbf{S}$. Further, this basis can be chosen so as to be orthonormal with respect to the positive definite quadratic form on $\mathbf{S}$ defined by the Riemannian metric on N. Thus, the elements of Ad F are simultaneously triangular and orthogonal matrices, hence must be of order two and diagonal, i.e. must commute.

We can now prove Theorem A. Let N be the submanifold obtained by Lemma 3.5. By Lemma 3.1 and Lemma 3.6, the subgroup S acting transitively on N is solvable with finite abelian isotropy group F all of whose elements are of order two. By Theorem B, the map Exp: $N^{\perp} \rightarrow M$ is a diffeomorphism. If F is the identity, $N^{\perp}$ is a vector bundle on S. If not, one has only to lift up the bundle via the covering $S \rightarrow S/F=N$ to finish the proof of Theorem A.

Finally, we prove a more special theorem. We are ultimately interested in seeing how more special information about M restricts the topological possibilities of the vector bundles, but results beyond Theorem 3.7 will be dealt with later.

Theorem 3.7. *Let M be a homogeneous Riemannian manifold of non-positive sectional curvature. Suppose further that $\pi_1(M)$ is cyclic. Then M is diffeomorphic to a homogeneous vector bundle over a circle.*

The proof depends on the following lemma due to S. Kobayashi [6]:

Lemma 3.8. *Let M be a homogeneous Riemannian manifold of non-positive sectional curvature. Let M' be the simply connected covering manifold of M', with π: $M' \rightarrow M$ the projection map. Let σ: $[0, \infty) \rightarrow M'$ be a geodesic in M' with $\pi(\sigma(0))=\pi(\sigma(1))$. Then: a) $\pi(\sigma(n))=\pi(\sigma(0))$ for each integer n, and b) There is a Killing vector field X on M such that $\pi_*(\sigma'(t))= =X(\sigma(t))$ for $0<t<\infty$. In particular, $\sigma_1(t)=\pi(\sigma(t))$ is a closed geodesic in M, i.e. satisfies: $\pi_1'(0)=\sigma_1'(1)$.*

Proof. Let T: $M' \to M'$ be the covering transformation such that $T(\sigma(0)) = T(\sigma(1))$. To prove a), we must show that $T^n(\sigma(0)) = \sigma(n)$ for each integer n. In terms of the metric on M', T has the following property: The distance from $T(x)$ to x is constant, for all $x \in M$. Consider the geodesic $t \to T(\sigma(t)) = \gamma(t)$, $0 \leqslant t \leqslant 1$. It goes from $\sigma(1)$ to $T(\sigma(1)$. To prove a), it suffices to show that $\gamma(0) = \sigma(1)$. Suppose otherwise. Then, the distance of $\sigma(\frac{1}{2})$ to $\gamma(\frac{1}{2})$ would be less than the sum of the lengths of $\sigma(t)$, $\frac{1}{2} \leqslant t \leqslant 1$, and $\gamma(t)$, $0 \leqslant t \leqslant 1$. But, $\gamma(\frac{1}{2}) = T(\sigma(\frac{1}{2})$, hence the distance between these points would be less than the distance of $\sigma(0)$ from $T(\sigma(0))$, contradiction. (This proof is well known.)

Then, $\sigma_1(t) = T(\sigma(t))$ satisfies: $\sigma_1(n) = \sigma_1(0)$ for each integer n. Since M is homogeneous, there is a Killing vector field X on M such that $X(\sigma_1(0)) = \sigma_1'(0)$. Consider the Jacobi vector field v: $t \to v(t) = X(\sigma_1(t)) - \sigma_1'(t)$ on σ_1. Since the curvature is non-negative, it must be zero, i.e. $X_\sigma(\sigma_1(t)) = \sigma_1'(t)$. For otherwise, since it is zero at $t = 0$, it would keep increasing in length, contradicting the fact that its values at integer t's must be bounded.

To prove Theorem 3.6 is now an application of Theorem B: N is the geodesic σ_1, the vector bundle in question is the normal bundle to the circle σ_1, and the group acting on σ_1 is the one-parameter group of isometries of M generated by X. It should be clear from the construction of σ_1 that the element of the fundamental group that it determines is the generator of $\pi_1(M)$, if the covering transformation T: $M' \to M'$ we used to construct it is the generator of the group of covering transformations of the covering $M' \to M$. Thus, the image of $\pi_1(N)$ in $\pi_1(M)$ is onto if σ_1 is chosen in this way.

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